Recitation 5

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Review

An $m \times n$ matrix A defines $A\colon \mathbb{R}^n \to \mathbb{R}^m$. The following are equivalent:

- A is **onto**;
- columns of A span $\mathbb{R}^m$;
- system $Ax = b$ has a solution (is consistent) for all b;
- **every row** of A has a pivotal position;

Once again, TFAE:

- A is **one-to-one**;
- $Ax = Ay$ implies that $x = y$ (i.e. A maps different vectors to different vectors);
- $Ax = 0$ has only the trivial solution (i.e. A doesn't kill any non-zero vectors);
- columns of A are linearly independent;
- **every column** of A is pivotal.

Standard matrix of T: see what T does to standard basis vectors $e_1, \dots, e_n$. Then $A = [T(e_1) \dots T(e_n)]$. So take e_1, apply T, it gives a vector in $\mathbb{R}^m$, that's your first column, et.c.

Subspace: H is a vector space, and $V \subset H$ is as subset, and you want to know if V is a **subspace**.
All you need: V is closed under $+$ and scalar multiplication. I.e., take two $v, u \in V$, compute $v + u$, is it again in V? Take any $v \in V$ and any scalar $c \in \mathbb{R}$, compute cv. Is it again in V? If both times "yes", then V is a subspace. If one answer is "no", then V is not a subspace.

Basis: vectors $v_1, \dots, v_n$ in a vector space V for a **basis** if and only if

- $v_1, \dots, v_n$ are independent;
- $v_1, \dots, v_n$ span the whole V.

Basis for $Col(A)$: pivotal columns <u>of A</u> form a basis of $Col(A)$ (but **not!** the columns of the reduced form of A).
Basis for $Span(v_1, \dots, v_n)$: it's the same as asking for a basis of $Col(A)$ with $A = (v_1 \dots v_n)$.
Basis for $Nul(A)$: you need to solve $Ax = 0$. So do that. Some variables will be free, some not. Express the solution as a vector, substituting non-free variables in terms of free ones. Then plug in 1 for one of free variables, put rest 0. This gives a vector. Do that for all free variables, get a bunch of vectors. This would be a basis for $Nul(A)$. Suppose for example you've got x_1, x_2 are non-free, x_3, x_4 are free, and suppose $x_1 = x_3 + x_4$, $x_2 = 2x_4$. There are two free variables, so there is two vectors in a basis for $Nul(A)$. So solutions look like

$$\begin{bmatrix} x_3 + x_4 \\ 2x_4 \\ x_3 \\ x_4 \end{bmatrix} \qquad \mapsto \qquad v_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, v_2 = \begin{bmatrix} 1 \\ 2 \\ 0 \\ 1 \end{bmatrix}$$

Put $x_3 = 1, x_4 = 0$ in the solution vector. This gives v_1 — the first basis vector for a basis. Put $x_3 = 0, x_4$, get v_2 — the second one. So a basis is $\{v_1, v_2\}$ as above.

Coordinates relative to a basis: Suppose $v_1, \dots, v_n$ is a basis of V, and b is any vector. To find **coordinates** of b you need to find scalars $x_1, \dots, x_n$ such that $x_1 v_1 + \dots + x_n v_n = b$. So really you **need to solve system of equations** $Ax = b$ with $A = (v_1 \dots v_n)$ (i.e. vectors v_i are columns of A).

Problems

Problem 1. Let $W \subset \mathbb{R}^3$ be the set of all vectors of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ satisfying the property $abc = 0$. Is W a subspace of $\mathbb{R}^3$?

Problem 2. Let $V \subset \mathbb{R}^3$ be the set of all vectors of the form $\begin{bmatrix} a \\ b \\ c \end{bmatrix}$ satisfying the property $a + b + c = 0$. Is V a subspace of $\mathbb{R}^3$?

Problem 3. Let

$$v_1 = \begin{bmatrix} 1 \\ -4 \\ 1 \\ 2 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 1 \\ 1 \\ 0 \\ -2 \end{bmatrix}, \quad v_3 = \begin{bmatrix} -3 \\ 0 \\ 0 \\ 3 \end{bmatrix}, \quad v_4 = \begin{bmatrix} -3 \\ -1 \\ 4 \\ 0 \end{bmatrix},$$

Are these vectors linearly independent? Do they span $\mathbb{R}^4$?
Find a basis for $Span(v_1, v_2, v_3, v_4)$.

Problem 4. Let $A = \begin{bmatrix} -2 & 3 & 4 \\ 4 & -2 & -6 \\ 1 & 0 & 0 \end{bmatrix}$. Do columns of A form a basis of $\mathbb{R}^3$?
Is A onto? Is A one-to-one?
What is the space $Col(A)$? Is A invertible? Find inverse of A.

Problem 5. Let $A = \begin{bmatrix} 1 & 2 & -1 \\ 3 & 7 & 1 \end{bmatrix}$. Find a basis for $Col(A)$ and for $Nul(A)$.

Problem 6. Let $A = [1\ 3\ -1]$. It defines a linear transformation $\mathbb{R}^3 \to \mathbb{R}$. Find bases for $Col(A)$ and $Nul(A)$.

Problem 7. Does the set of vectors $\{v_1 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}, v_2 = \begin{bmatrix} 0 \\ 2 \end{bmatrix}\}$ form a basis of $\mathbb{R}^2$?
Compute coordinates of the vector $x = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$ relative to the basis $\{v_1, v_2\}$.

Problem 8. Find a basis $\mathcal{B}$ for the space V from Problem 2 (hint: see Problem 6).
Compute the coordinates $[x]_{\mathcal{B}}$ of the vector $x = \begin{bmatrix} -3 \\ 4 \\ -1 \end{bmatrix}$ is this basis. How many entries does $[x]_{\mathcal{B}}$ have?

Problem 9. Define a transformation $T \colon \mathbb{P}_3 \to \mathbb{P}_2$ sending each polynomial $\mathbf{p}$ to its derivative $\mathbf{p}'$, i.e. $T(\mathbf{p}) = \mathbf{p}'$. For example, $T(x^2 - x) = 2x - 1$. Find the kernel of T and describe the range of T.

Problem 10. For the following statements, mark them "true" or "false" and explain your answer.

1. A single vector by itself is linearly dependent.
2. If $H = Span\{b_1, \ldots, b_n\}$ then $\{b_1, \ldots, b_n\}$ is a basis for H.
3. If $\mathbb{R}^n = Span\{b_1, \ldots, b_n\}$ then $\{b_1, \ldots, b_n\}$ is a basis for $\mathbb{R}^n$.
4. The columns of an invertible $n \times n$ matrix A form a basis of $\mathbb{R}^n$.
5. A basis is a spanning set that is as large as possible.
6. If B is an echelon form of a matrix A, then columns of B form a basis for $Col(A)$.

Problem 11. Suppose v_1, v_2, v_3 are **dependent** vectors in $\mathbb{R}^3$, and suppose $b \in Span(v_1, v_2, v_3)$ is a vector in their span. Prove that the vector b can be expressed as $b = x_1v_1 + x_2v_2 + x_3v_3$ in more than one way.
Do the same but using a "pivot-involving" argument.